b. Hind angles of thorax moderately prominent:

·34—·37. Antennæ, tip of tibiæ, tarsi and trunk dark brown; thorax scarcely punctulate; elytra scarcely costate. Kansas. stygicornis *Say.*

c. Hind angles of thorax very slightly prominent:

·27—·35. Elytra scarcely costate; antennæ and abdomen not infuscated. Middle and Western States, Kansas. *cordicollis*‡ Lec. . rejectus *Lec.*

F. Smaller species; thorax longer than wide, convex, constricted behind, hind angles prominent; elytra, humeri distinct.

a. Epipleuræ as usual of the color of the elytra:

·29. Elytra oblong, scarcely dilated behind; abdomen not infuscated. Middle States. *Aptinus janth.* Dej. . . . janthinipennis *Lec.*

·36. Elytra broader quadrate, dilated behind, slightly costate; trunk infuscated. Southern and Western States. . . quadripennis *Dej.*

·18—·25. Elytra dilated behind, not costate; abdomen dark brown. medius *Lec.*

b. Epipleuræ pale testaceous;

·33—·39. Thorax less narrowed behind than usual; sides of metathorax and abdomen, and knees dark. Southern States and Arizona, as far as Colorado River. . . . . lateralis *Dej.*

G. Small species; thorax longer than wide, hind angles not prominent; elytra wider behind, humeri indistinct.

·28—·36. Thorax very broadly rounded on the sides in front; sides of abdomen dark. Middle and Southern States. *cephalotes*‡ Lec. ovipennis *Lec.*

a. Thorax more strongly rounded on the sides in front:

·30—·36. Abdomen dark brown. Middle, Southern and Western States. *patruelis* Lec. . . . . conformis *Dej.*

·21. Abdomen not infuscated. Middle States. . . pumilio *Lec.*

Contributions to Organic Morphology:—Containing the mathematical imitation of the egg of PLANORBIS CORNEUS and of EPIORNIS; and upon the resemblances between Mathematical, Acoustic, Electric, Optical and Organic Figures; with historical and other remarks.

BY JOHN WARNER, A. M.

PREFACE.

In a work published several years ago, I endeavored to make some contribution to the knowledge of Organic Morphology.* Among other matter, the work contained the results of investigations made to determine the coincidence in form between sections of hen's eggs and a curve there proposed. In the present paper, it is designed to extend these investigations, and to notice some other subjects of interest. Some use will be made both of original and selected matter contained in the work referred to, and other citations will be added, which seem appropriate in treating a branch of science not yet possessing a classified and independent literature.

MORPHOLOGICAL LITERATURE.

Numerous authorities might be cited, bearing upon the general subject of our research, and containing information and suggestions worthy of study;

* Studies in Organic Morphology, by John Warner. J. B. Lippincott & Co., Philadelphia, 1857.

but extensive reference to these authorities would be inconsistent with our present limits: many of them, either directly or indirectly, advocate the possibility of a mathematical explanation of the cause of organic forms.

Professor Bronn* considers that there is an inconsistency in supposing the organic world alone to be derived from a direct act of creation, whilst all the rest is born and perishes from the effect of general forces eternally immanent in matter. He concludes that all species of animals and vegetables were originally created by a natural force, at present unknown—that they do not owe their origin to a successive transformation of a few primitive forms—and that this force held a most intimate and necessary relation to the forces and events which have controlled the development of the surface of the globe. He thinks that such a hypothetical force would be in entire harmony with the whole economy of nature, and that the hypothesis would not only permit the belief in a Creator presiding over the development of organic nature by means of an intermediate force, but that this conception is more sublime than the idea of a direct supervision, by the Creator, of the succession of plants and animals. Professor Bronn also considers the fundamental form of a plant to be that of an egg placed upright. Investigation of the relation between natural and mathematical ovoid forms might furnish a test for the correctness of this idea, or, if it is well founded, assist in explaining its application.

Some mathematical writers treat as an evident proposition the ultimate connection between mathematics and the explanation of natural processes.† Fechner undoubtedly encourages this idea, and even proposes, more or less definitely, the adoption of a mathematical classification in physiognomy, craniology, and ethnology.‡

Lotze, on the other hand, takes the opposite extreme. In one of his more skeptical passages he compares the attempt to discover the laws of organiza-

* Essai d'une Réponse à la question de Prix, &c. Comptes Rendus, vol. 51, p. 511.

† The principles of mechanics must be of the greatest importance for all branches of natural science, (as Aristotle was aware,) because, according to our conception of the changes of the material world, they must be referred to motion. Dr. H. Burhenne, Grundriss der Hoeheren Analysis, Cassel, 1849, p. 84.

Dr. Zeising, and others whom we have cited, refer at length to the works of Pythagoras, Plato, and Aristotle, in order to show that the ancients regarded numbers as in some mysterious sense the principia of the universe. The Pythagorean quaternary, as improved by Plato, consists of the celestial numbers 1, 3, 7, 9, of which the sum is 20, and of the terrestrial series 2, 4, 6, 8, whose sum is likewise 20. These two together make the sacred quaternary 40. The number 5, which is not in the quaternary, but is the middle of the whole series from 1 to 9, represents the *Nous*, or supreme intelligence. According to Montucla, these numbers and the idea of their mystic importance were derived from the Egyptians. The ancient Chinese also venerated the Pythagorean quaternary, and ascribed its invention to the emperor Fo-hi (2900 B. C.) Fo-hi was the inventor of the binary arithmetic, of which he left the notation in the Cova, or Figure of Eight. M. Huc relates that the Chinese still venerate a mysterious book, called the Book of Changes, *y-King*. The meaning of this book has long been lost. From M. Huc's description of the 64 whole and broken lines of this book, and from Leibnitz's description and interpretation of the Cova, I have little doubt that the *y-King* pertains to the arithmetical system recorded in the Cova. The tradition of the Chinese, that the *y-King* is capable of explaining all things, may, therefore, indicate that the ancient Chinese were not unaware of the importance of number in the order of the universe, and that their sages had conceived the idea of a mathematical explanation of Nature, as clearly as such an idea could be conceived in advance of the science of physical mathematics: possibly they progressed no further than to incorporate the Cova in their religious mysteries. Montucla, Histoire des Mathématiques, vol. i. p. 122. Chinese Empire, by M Huc, London, 1855, vol. i. p. 124. Leibnitz, Mémoire de l'Acad. Française, vol. xviii. 1703, p. 85. Dr. H. Burhenne, Grundriss der Hocheren Analysis, Cassel, 1849, p. 84.

‡ Ueber die Mathematische Behandlung Organischer Gestalten und Processe. Verh. d. Koenl. Saechs. Gesellsch. Mathematisch-Physische Kl., Jarhgang 1849.

Mr. Hay has published a method for defining geometrically the shape of the human head and the proportion of its parts. The method is founded on a system of triangles, of which the angles have certain ratios manifested in the vibrations of musical strings. See D. R. Hay on the Beau-Ideal Head of Ancient Greek Art. Trans. Soc. of Arts, vol. i. part 2, New Series, 1847–8.

The same author has written several works on the Beautiful in Form. His Natural Principles of Beauty (London and Edinburgh, 1852) gives a concise explanation of his geometrical construction of the human figure. The same subject is differently treated by Dr. A. Zeising, Neue Lehre von den Proportionen des Menschlichen Koerpers; Leipzig, 1854. The student of Morphology will be interested in comparing with these works, Die aus der Arithmetic und Geometrie herausgeholten Gruende zur Menschlichen Proportion; Georg Lichtenstegcr, Nuremberg, 1746.

tion by the study of organic forms, to the endeavor to decipher the principle or purpose of a complicated machine by the contemplation of its shadow.* He discourages the notion that the shape of the egg is susceptible of a mathematical explanation. The form of the egg, he considers, is not the immediate product of a formative tendency, but the mechanical result of a twisting action of the oviduct, and gives as little hope of an explanation of the forming forces as, for example, the shape of a top does of comprehending the law of formation of the person who turned it.†

Meckel‡ accounts for the form of the egg in a similar manner. He cites Thienemann to show that when the egg is forced rapidly through the oviduct, in consequence of persistently chasing the hen, the egg is then deformed, being greatly elongated and without a hard shell. He also alludes to the experiments of M. St. Hilaire in proof of the fact that hen's eggs placed vertically during incubation either do not come to development or else produce monsters. On the whole, he appears to be of opiuion that the form of the egg may not only have a mechanical origin, but may be important as a mechanical means in determining the form of the embryo.

OF MATHEMATICAL OVOIDS.

Fechner adopts the oval of Descartes, proposed by Steiner, as the true representative of the form of the egg.§ The elliptic spheroid he considers to be a rough approximation to the true form; but M. St. Hilaire states that out of six eggs of the Epiornis, sent to France, five were nearly true ellipsoids.‖ The other had a large and a small end. We shall now consider particularly the curve proposed by ourself to represent the longitudinal section of an egg. This curve belongs under a general formula which includes the ellipse. We shall principally consider a curve having an obtuse and an acute end, and which may be called the *hyper-ellipse*, and the solid generated by its revolution, the *hyper-ellipsoid.*¶

Construction of the hyper-ellipse.—Measure the length and thickness of the egg. Draw (Fig. 1, Plate 1) A B, H D, each equal to the length of the egg, and bisecting each other at right angles in C. Make D K equal to the half-thickness of the egg, and on H K describe a semicircle cutting A B in F. Then A B is the axis of the hyper-ellipse, and F is the focus.

Construct an ellipse (Fig. 2) with the semi-axes F A, F B equal respectively to the same distances in Fig. 1, and draw any radius vector F P.

In Fig. 3 draw B F, F A, as in Fig. 1, and make the angle A F P equal to *twice* the angle A F P of Fig. 2; also make F P equal to the same in Fig. 2. The point P is then a point of the hyper-ellipse. In a similar manner any required number of points may be found, and the curve traced through them by hand. Instead of beginning the construction at A, we may commence at B, making the angle B F P equal to twice the same of Fig. 2, and the radius F P the same.

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* Allgemeine Physiologie des Koerperlichen Lebens. Leipzig. 1851, p. 328.

† Ibid. p. 335. The labors of Hanstein and Wright in investigating the law of phyllotaxis—although they do not prove mathematically the cause of phyllotaxis, but rather pertain to its teleological significance—appear to me to contain remarkable applications of mathematics to the study of Organic Morphology, and to take much from the general force of Lotze's criticism. Hanstein ueber den Zusammenhang der Blattstelluug mit dem Bau des dikotylen Holzringes. Monatsber. d. Koenl. Preuss. Ak. d. Wiss., Berlin, 1857, p. 105. Wright on the most thorough, uniform distribution of points about an Axis. Mathematical Monthly, April, 1859.

‡ Die Bildung der fuer partielle Furchung bestimmten Eier der Voegel, &c. Zeitschr. f. Wiss. Zoologie, vol. 3, 1851, p. 432.

§ We may refer the reader to Mr. Hay's Principles of Symmetrical Beauty, and to Purdie on Form and Sound (Edinburgh, 1859), for information concerning the composite ellipse—a figure which seems to offer or to suggest means for closely imitating the forms of various eggs.

‖ Note sur des ossements et des œufs trouvés à Madagascar, dans des alluvions modernes, et provenant d'un oiseau gigantesque; par M. Isidore Geoffroy-Saint-Hilaire. Comptes Rendus, vols. xxxii. p. 101; xxxix. p. 833; xlii. p. 315, and xliii. p. 518.

¶ This curve may be termed the hyper-ellipse, because its radius vector is a power of the radius vector of an ellipse, taken from the focus, or because its radius is derived from the ellipse as in the following construction.

The curve can also be constructed by drawing F P from any assumed pole at F, at any angle with an assumed axis A B. Then, knowing the length and thickness of the egg and the angle A F P, the length F P can be calculated by the aid of a proper formula, hereafter given, and transferred to the drawing.

COMPARISON OF EGGS WITH THE HYPER-ELLIPSOID.

Egg of Planorbis corneus.—The example just given to show the construction of the hyper-ellipse (Fig. 3) presents a good imitation of the magnified drawing of an egg of Planorbis corneus.* It is remarkable that the focus F of the theoretical egg falls, as nearly as can be readily observed, in the centre of the vitellus, according to the engraving given by Jacquemin. The magnified egg of the engraving measures, say, length 1·63 inches, thickness 1·31. The distance B F is found by calculation ·45+.

Egg of Epiornis.—The cast of the egg of Epiornis belonging to the Academy of Natural Sciences in this city is, I doubt not, from the pointed egg described by M. St. Hilaire.† A longitudinal section of this cast was obtained by cutting a templet to fit closely around it, then tracing the form of the egg from the templet. The first section thus obtained was not quite symmetrical with respect to the long axis. A second section, taken on a plane at right angles to the plane of the first, was more nearly symmetrical. The cross-section of the cast measured so nearly circular, that the small difference in the diameter of the sections taken may be disregarded. For the purposes of calculation we have taken M. St. Hilaire's dimensions reduced to inches,—length 12·756, thickness 8·859.

Fig. 4 represents the theoretical egg on a scale of one-fourth. The following tables exhibit the dimensions of the two real sections and of the theoretical section. Each real section is divided by the axis into two parts not entirely symmetrical. The radii vectores of each part are given for various polar angles. The distance from the obtuse end of the egg to the line of greatest thickness is not involved in the construction of the theoretical ovoid. This distance as measured on the egg should be compared with the same as found by construction. In the tables it is designated by $B' + \rho \cos \gamma$.

TABLE I.

MEASURED VALUES FOR REAL EGG.

Designation.		Values of the radius vector for various polar angles.										
		0°	10°	20°	30°	40°	γ	70°	90°	110°	130°	180°
1st Real Section.	Part 1.	9·90	9·56	8·64	7·66	6·67	5·49	4·59	3·76	3·23	2·98	2·85
	Part 2.	9·90	9·37	8·42	7·47	6·60	5·49	4·65	3·85	3·31	3·02	2·85
2d Real Section.	Part 1.	9·90	9·52	8·62	7·56	6·63	5·49	4·62	3·82	3·32	3·07	2·85
	Part 2.	9·90	9·43	8·52	7·56	6.65	5·49	4·59	3·79	3·25	2·98	2·85

NOTE.—The distance from the large end of the egg to the greatest cross-section $= B' + \rho \cos \gamma = 6$ inches. The radius for a polar angle of 56° 34′ is 5·39, average of four dimensions.

* Mémoire sur l'Histoire du Planorbis corneus, par Emile Jacquemin. Nova Acta Acad. C. Leopold. vol. xviii. p. 638.

† The agreement in size is satisfactory, and the egg is marked with the name of Dr. Warren, who relates that a cast of the pointed egg was presented to him. (Fossil Impressions, &c., Boston, 1854.) The length which he gives for the egg is incorrect. The mistake probably arose from a typographical error, which is corrected in Comptes Rendus, vol. xl. p. 519.

TABLE II.

CALCULATED VALUES FOR THEORETICAL EGG.

Designation.		Values of the radius vector for various polar angles.										
		0°	10°	20°	30°	40°	γ	70°	90°	110°	130°	180°
Ideal Section.	For both parts.	9·90	9·51	8·58	7·51	6·54	5·32	4·60	3·88	3·41	3·02	2·85

NOTE.—The length and thickness of the egg are the same for both tables, viz.: length 12·76—; thickness 8·86. The polar angle at the point of greatest thickness is designated by γ, but has not the same value in each table, being a measured value in Table I. and a calculated value (56° 34′) in Table II. The distance $B' + \rho \cos \gamma$ for this angle is, by calculation, 5·79.

By these tables we perceive that the first part of the first section agrees very closely, from the apex to the widest part, with the theoretical curve. The second part is less satisfactory. The average real section would nowhere differ from the theoretical curve by much more than the thickness of the egg-shell (about 12-100 inch).

ON THE SIGNIFICATION OF THESE COMPARISONS.

Position of the Vitellus.—A belief in the adaptability of polar formulæ to explain some processes of nature was somewhat vaguely expressed by Grandus. James Bernoulli seems to have been strongly, though vaguely, impressed with the idea of an important meaning in the logarithmic spiral. Moseley and Naumann have demonstrated its existence in several shells, and similar results have since been obtained.* Possibly the mechanical properties of this spiral are involved among the causes which give the first direction to the windings of shells. Naumann suggests that all spirally wound conchylia may begin with a logarithmic spiral. The law of the growth of the animal may then, perhaps, be subsequently instrumental in determining the form of the shell.† Lotze says, in discussing the mathematical explanation of organic forms, that in pure mathematics it is not absolutely essential that the origin of co-ordinates be in any particular place, but where an explanation of the nature of phenomena is required, the origin must be taken where, in the Real, the centre of emanation of action resides; the direction and connection of the co-ordinates must correspond with those of the operating forces.‡

The position of the vitellus in the pole of the theoretical egg may, possibly, exhibit that coincidence of mathematical and organic system which is intended by Lotze. It would be desirable to have correct observations of the positions of different parts within the egg, in order to attempt the discovery

* Grandus, A Collection of Geometrical Flowers: Abridged Phil. Trans., vol. vi. p. 67, 1723. Bernoulli, Leipzig Acts., 1692. Moseley, Phil. Trans., 1838. Naumann, Abh. d. Jablonowskischen Gesellsch., Leipzig, 1846. Also Abh. d. Math. Phys. Kl. d. K. S. Gesellsch., Leipzig, 1852. Sandberger, ueber die Spiralen von Ammonites, &c., Zeitschr. d. Deutsch. Geol. Gesellsch., vol. x. 1858, p. 446.

† I am not aware to what extent the views of Moseley concerning the growth of shells have been adopted by naturalists. Naumann, as far as he expresses any opinion, seems to agree with Moseley on this subject. To me, Moseley's explanation of the growth of shells, and of the manner in which their mathematical properties suit the life and growth of the animal, appear very interesting and important, and, did our limits permit, would well deserve to be noticed here at length. From the paper of Sandberger's just cited, I am led to believe that the determination of the equations of the windings of shells is now recognized by scientific conchologists as a valuable descriptive method.

‡ Physiologie, p. 330.

whether any of these parts have a position remarkable in a mathematical point of view, and which might, therefore, possibly suggest something important in regard to further researches.*

Cubical contents.—The volume of the hyper-ellipsoid is equal to the solidity of a prolate ellipsoid having the same length and thickness.† Hence it appears that a definite quantity of material fit for the composition of an egg might, considered geometrically without regard to other conditions, take the form of either an ellipsoid or hyper-ellipsoid egg; the length and the thickness being the same in each case. It seems probable that either form might suit the structure of those parts of the bird which anatomists believe to be most directly concerned in giving shape to the egg. I do not certainly know whether the same hen can lay eggs of both forms. Among several hyper-ellipsoid eggs, said to be from the same hen, was found one which most observers would probably consider ellipsoidal. A gentleman who once took much interest in the breeding of fowls states that, whilst engaged in this pursuit, he was able from the appearance of the eggs, but not judging alone by the shape and size, to recognize with considerable certainty the eggs of particular birds and of particular breeds. In his opinion, the eggs of the same hen would appear, to ordinary observation, to be of nearly the same size and shape: sometimes, however, an unusually large egg containing two yolks will be produced. My limited observation is, in general, in favor of the supposition of uniformity of size and shape among the eggs of the same individual. I have, besides measuring some hen's eggs, carefully inspected the eggs found in several nests of wild birds.

Standard of comparison for shape.—As far as I am aware, no mathematical standard of comparison for the shape of eggs has been fixed. Thus, for example, if we had an egg intermediate, as the term would generally be applied, between the ideal form above found for the egg of Epiornis, and a true ellipsoid having the same diameters, it would be left to the judgment alone to decide which ideal form should be preferred as a representative of the egg.

M. St. Hilaire does not give measurements to show the agreement between the ellipsoidal eggs of Epiornis and true ellipsoids. Of some of them, he says their resemblance to each other was so great that one might have readily been mistaken for the other. From this description I doubt whether these eggs resembled ellipsoids more nearly than the present egg of Epiornis resembles the hyper-ellipsoid.‡

RESEMBLANCES BETWEEN MATHEMATICAL, ACOUSTIC, ELECTRIC, OPTICAL AND ORGANIC FIGURES.

The mathematical laws of the propagation of light are shown to be particular cases of the more general laws of vibratory motion in any elastic medium composed of attracting and repelling molecules.§ It would, therefore, seem that forms similar to those shown in the polarization of light, and in other

* I am not informed how far naturalists have considered this subject. Some experiments of my own, made on hen's eggs, in order to ascertain the relation between the size of the yolk and other dimensions of the egg, and also whether the centre of the yolk more nearly coincides with the centre of gravity of the egg or with the centre of the axis, resulted in gaining some preliminary experience in the method of observation, but did not establish any thing certain in regard to the object of research.

† The rule for computation is, Multiply the square of the thickness by the length, and the product by ·5236. The result is the solidity.

‡ Since writing the above I have seen, in the collection of the Smithsonian Institution, the cast of an egg of Epiornis (the egg sent from Madagascar in 1850), but have not had an opportunity of closely examining it. The date indicates that it is from a cast of the ellipsoidal egg described by M. St. Hilaire in his first memoir, and which accompanied the egg we have endeavored to imitate in Fig. 4. Without a careful measurement of the cast, it would, in my opinion, be unsafe to conclude that the egg in question is more nearly ellipsoidal than its fellow is hyper-ellipsoidal.

§ Encyclopedia Britannica, Boston ed., art. Optics, p. 546.

optical experiments, might result from the vibrations of other substances which in their vibrations may follow the same or similar laws. This may possibly be the reason of some resemblances of the kind we shall now consider.

Construction of the Hyperaster.—Figures resembling star-fish may be derived from the ellipse by a construction similar to that given for Fig. 3. Both constructions can be included under a general mathematical formula.* To construct the hyperaster with five points, make the ellipse (Fig. 5) with the semi-axis F B equal to the short radius F B (Fig. 6) of the star, and with the longer semi-axis F A equal to the long radius F A of the star. Then, beginning at B, proceed as for the construction of Fig. 3, *except* that the angle B F P of Fig. 6 is to be always taken equal to *two-fifths* of B F P in Fig. 5. When the radius F P of Fig. 5 has passed through a revolution of 90°, it will coincide with F A, and P will then fall upon A. During the same time, the radius F P of Fig. 6 will pass over two-fifths of 90°, or 36°, and will reach A. When the radius of Fig. 5 reaches F C, it will have passed over 180°, and in Fig. 6 the radius, then at C, will have passed over two-fifths of 180°, or 72°, which is the fifth part of the circumference of the circle. The arm B A C F of the star is therefore derived from the semi-ellipse B A C. A repetition of the same process will derive the next arm of the star from the semi-ellipse C D B; and so on, until the five arms of the star are completed.† By means of this construction, star-fish or other organic bodies resembling them can be imitated. Returning to Figs. 2 and 3, it will be observed that, starting at B, the entire Fig. 3 is generated from the semi-ellipse B A C, in the same way that the arm B A C F of Fig. 6 is derived from the semi-ellipse B A C, Fig. 5. Viewed in this manner, the egg, Fig. 3, appears a one-armed star-fish. Whether or not this conception may have any significance in nature, it appears remarkable to find two different organic forms thus classed under the same mathematical formula. Some of the figures known as acoustic figures, produced by the vibration of elastic plates, can also be imitated. Figures resembling Fig. 6 are given by Chladni in his treatise. Possibly the acoustic figures might be produced on a scale sufficiently large to test their agreement with the mathematical figures, by measurement; and hence it could be, perhaps, determined whether these truly represent the former.

Interesting resemblances can be traced between the optical and acoustic figures, and between these and curves similar in their construction to those we have described, if not always precisely of the same construction. The following is of the same general construction as the previous. By taking the ellipse Fig. 2, and making the angle B F P in the derived figure always equal to *one-half* of the same in the ellipse, we derive a curve similar to Fig. 7. Figs. 8 and 9 represent an optical figure and an organic form, having a resemblance to this.‡

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* Studies in Organic Morphology, Formula 2, p. 32. We propose to call the curve whose equation is $\rho = \frac{p}{1 - e \cos k\theta}$ the *elliptoaster*, because the equation resembles that of the ellipse, and the curve itself may represent a star. The name *hyperaster* may be given to the curve whose radius is a power or root of the radius of the elliptoaster.

† In actual constructions, it will be sufficient to derive one arm of the star, and then, by means of tracing-paper, to dispose five such arms around the centre F. Stars of any desired number of points may be thus constructed; the angle B F P of the star must be to the angle B F P of the ellipse as the number 2 is to the number of points in the star.

‡ See Encyclopedia Britannica, Boston ed., art. Optics, p. 672, for Fig. 9. For Fig. 8 see Zeitschr. für Wiss. Zoologie, Leipzig, 1854, vol. v. Plate XIV. Fig. 34. These resemblances could be followed to a greater extent. The writer has collected many drawings of mathematical lines, organic objects, optical, acoustic, and electric figures, but must omit further notice of them on the present occasion. By large collections of this kind, and by diligent comparison of their materials, something may, perhaps, be elicited which will establish a reliable foundation for the study of Organic Morphology as a mathematical science.

An electrical figure having a strong resemblance to an egg may be seen on Plate III. of Lichtenberg's figures.*

M. Cornay considers electricity to be the radical universal generator. He endeavors to illustrate this idea by comparing positions assumed by electrically charged needles to the positions of parts of plants and animals. For this purpose he has numerous engravings.† His description of the circulation of the electric fluid, and of the effect of it in producing the nervation of leaves and the spiral arrangement of leaves around the trunk of the plant, reminds us of similar suggestions of Grandus to account for the disposition of the petals of a flower. But M. Cornay's resort to experiment to test his opinions is an important step in the right direction, for which he deserves the thanks of morphologists, although as yet his experiments cannot be considered conclusive proof of the correctness of his views.

EXPLANATION OF THE PREVIOUS CONSTRUCTIONS—CUBATURE OF THE HYPER-ELLIPSOID.‡

Construction of the Hyper-ellipse and Hyperaster.—Let (Fig. 3) the axis A B, or length of the egg, $= 2\,a$, and the greatest double ordinate, or thickness of the egg, $= 2\,m$. We have shown, in our work already referred to, that $\mathrm{F\,A} = a + \sqrt{a\,(a - m)}$ and $\mathrm{F\,B} = a - \sqrt{a\,(a - m)}$: it is now required to find these distances by construction. By the construction given for Fig. 1, $\mathrm{D\,K} = m$, $\mathrm{C\,D} = a$, therefore $\mathrm{C\,K} = a - m$. But, by Geometry, C F is a mean proportional between C H and C K, that is, between a and $a - m$. Hence $\mathrm{C\,F} = \sqrt{a\,(a - m)}$; whence $\mathrm{F\,A} = \mathrm{C\,A} + \mathrm{C\,F} = a + \sqrt{a\,(a - m)}$, and $\mathrm{F\,B} = \mathrm{C\,B} - \mathrm{C\,F} = a - \sqrt{a\,(a - m)}$; which was required.

We have further shown that the radius vector of the hyper-ellipse is equal to the radius vector of an ellipse referred to the centre, and in which the polar angle is one-half that of the hyper-ellipse. This is the ellipse shown in Fig. 2, and hence the construction before given for Fig. 3 is evident. By referring to our work, it will be seen that the hyperaster, Figs. 6 and 7, may also be constructed from an ellipse in a similar manner, taking care that their polar

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* Commentationes Societatis, &c., Goettingen, 1778, vol. i. For a curious resemblance to a tree, produced by the action of lightning, see Mr. Charles Tomlinson on Lightning Figures, Edinburgh New Phil. Journal, vol. xiv. No. 2, Oct. 1861, and vol. xv. No. 1, Jan. 1862.

† Principes de Physiologie et Eléments de Morphogénie Générale, par J. E. Cornay (de Rochefort), Paris, 1853, pp. 112, 191, 212—215. M. Cornay has labored earnestly and industriously to promote the knowledge of Morphology. Some important propositions which he confidently assumes appear to us still to want satisfactory proof. Thus, for example, because the shape of an insect agrees with the outline of a cluster of electrified needles, he appears to be satisfied that he has found in the action of electricity, or of some hypothetical fluid, the true cause of the organic form.

‡ For certain formulæ which will be necessary in this and the following investigations, see Studies in Organic Morphology, pp. 32, 33, 40, 41. The curves now to be discussed belong to the general form

$$\rho = \left(\frac{p}{1 - e\cos k\,\theta}\right)^n;$$

wherein p is the semi-parameter, and e the eccentricity, of an ellipse. For the hyper-ellipse, $k = 1$, $n = \frac{1}{2}$. In Fig. 5, $k = 5$, $n = \frac{1}{2}$. In Fig. 7, $k = 4$, $n = \frac{1}{2}$. The equation $\rho = \frac{p}{1 - e\cos k\,\theta}$ represents an immovable orbit substituted for an elliptical orbit revolving about its focus, Prop. XLIII. B. I., Newton's Principia, and Wright's Commentary on the Principia, London, 1828, vol. ii. p. 245. Curves of the sort in question may be produced by revolving an ellipse, under various conditions, whilst a describing point revolves in the ellipse. Some years since, I exhibited to the Pottsville Scientific Association a mechanical arrangement for producing such figures. Suardis's Geometric Pen is also an interesting instrument for describing curves. The joints of the pen remind us of the joints in the limbs of animals, and furnish a hint as to a mathematical conception of the motion of the limbs. See Adams's Geometrical and Graphical Essays, London, 1813, p. 151.

angle has the proper proportion to that of the ellipse, in order to derive the number of arms or rays desired.*

Cubature of the Hyper-ellipsoid.—Let F, Fig. 10, be the pole, P M an infinitesimal arc, and P F M an elementary triangle of any plane curve, referred to the axis F N, which is also the axis of revolution for the solid.

The centre of gravity, G, of the elementary triangle P F M, is on D G, drawn parallel to the side P M, and so situated that F D is two-thirds of the radius F P, $= \frac{2}{3}\rho$. When the side P M vanishes, F P will coincide with F M, and the distance from F to G will then equal F D $= \frac{2}{3}\rho$, and the angles M F N, G F N, P F N will all be equal, and each $= \theta$. The distance G N will be F D $\sin\theta = \frac{2}{3}\rho\sin\theta$; and the distance described by G during a revolution of the elementary triangle P F M about the axis F N will be F D $2\pi = \frac{4}{3}\pi\rho \sin\theta$. The area of the elementary triangle is, however, $\frac{1}{2}\rho^2\, d\theta$, and the solidity of the conical sheet generated by a revolution of P F M, which is the differential of the solid of revolution, will be, by Guldin's Formula,

$$dV = \tfrac{4}{3}\pi\rho\sin\theta .\ \tfrac{1}{2}\rho^2\, d\theta = \tfrac{2}{3}\pi\rho^3\sin\theta\, d\theta \ldots\ldots (1)$$

In the present case this becomes

$$dV = \tfrac{2}{3}\pi\frac{p^{\frac{3}{2}}\sin\theta}{(1-e\cos\theta)^{\frac{3}{2}}}\,d\theta = -\tfrac{4}{3}\pi\frac{p}{e}\left(\frac{-p^{\frac{1}{2}}e\sin\theta}{2(1-e\cos\theta)^{\frac{3}{2}}}\right)d\theta:$$

in which last $\dfrac{-p^{\frac{1}{2}}e\sin\theta}{2(1-e\cos\theta)^{\frac{3}{2}}}\,d\theta$ is the differential of the radius vector ρ; so that we have, by substitution, for the solidity of the whole hyper-ellipsoid,

$$V = \int_{\theta=0}^{\theta=\pi} -\tfrac{4}{3}\pi.\frac{p}{e}\,d\rho \ldots\ldots (2)$$

If the radius for $\theta = 0$ be denoted by ρ' and the radius for $\theta = \pi$ by ρ'', this equation gives

$$V = \tfrac{4}{3}\pi\frac{p}{e}(\rho' - \rho'') \ldots\ldots (3)$$

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* We may here call attention to the fact that the radius vector of the hyper-ellipse, for the extremity of the greatest ordinate, is $\rho = \sqrt{a\, m}$, that is, this radius is a mean proportional between the half-length and half-width of the figure. This is interesting because Dr. Zeising adopts the mean proportion as a general morphological law; but this proportion of itself cannot be satisfactory: we require some rule for knowing what objects or parts of objects are to be thus compared. As long as no such rule exists, the comparisons may often seem arbitrary. Dr. Zeising proposes, for the egg-curve, to divide the length into two parts, say a' the greater and m' the lesser; then m' will also represent the half-thickness, and we shall have the proportion $a' + m' : a' :: a' : m'$, whence $a' = \sqrt{m'(a' + m')}$. It is not, however, shown by him that this mean proportion is necessarily more significant in Morphology than $\rho = \sqrt{a\, m}$ above mentioned (Neue Lehre, p. 228). Dr. Zeising's application (Neue Lehre, p. 361) of the extreme and mean ratio, or golden section, to the division of the circle in phyllotaxis, has received a remarkable confirmation as a law of nature, by the labors of Hanstein and Wright, before cited. His application of this ratio to the relations of the planetary system seem to me worthy of close study; but proof is required of a similar significance of this ratio in astronomy and in botany, before we can assume that there is an entire identity between the laws which regulate both the planetary and the phyllotactic systems. (Neue Lehre, p. 327. Normalverhältniss, &c., Leipzig, 1856, pp. 2, 45.)

But we have seen (Fig. 3) that $\rho' = FA = a + \sqrt{a\,(a-m)}$ and $\rho'' = FB = a - \sqrt{a\,(a-m)}$; hence $\rho' - \rho'' = 2\sqrt{a\,(a-m)}$. Further, in our former work, we have shown that $p = \frac{m^2\,a}{2\,a - m}$, and $e = 2\,\frac{\sqrt{a\,(a-m)}}{2\,a-m}$; hence $\frac{p}{e} = \frac{1}{2}\,\frac{m^2\,a}{\sqrt{a\,(a-m)}}$. Substituting these values of $\rho' - \rho''$ and of $\frac{p}{e}$ in equation (3), there results

$$V = \tfrac{4}{3}\,\pi\,m^2\,a \qquad (4)$$

which is the volume of a prolate ellipsoid whose semi-transverse axis is a and semi-conjugate m.

The further discussion of the hyper-ellipse has led us to some interesting formulæ, which, whether this curve is really important in Morphology or not, appear worthy of attention in a mathematical point of view. These formulæ we hope to present in a subsequent paper.

CONCLUDING REMARKS.

The coincidence in form between organic outlines and mathematical figures is a subject of difficult interpretation. It will, therefore, be sufficient for me, without expressing a confident opinion as to the meaning of such resemblances, to say that my study of the subject has induced the belief that all the resemblances of this kind which have been found are not mere accidental coincidences, but that some of them are the results of a mathematical arrangement in nature. The reason or the fitness of this arrangement, I am inclined to think, is explicable mathematically, at least to a greater extent than has been generally supposed, and the search for such an explanation I conceive to be a legitimate effort of science. I would, therefore, encourage the collection of drawings of organic objects and of mathematical figures, together with other materials for study, in order to combine facts as they appear, and prepare for the discussion of definite questions which may be suggested. Works or memoirs devoted to the measurement of organic products, constituents, and combinations, or of organic functions, as measured by phenomena of production, of motion or duration, or by the evolution or abstraction of force, may probably be useful in furnishing data for the study of the cause of organic forms. But we shall not prescribe rules in this respect. Of late years many researches of this kind have been made, and in several cases by naturalists or physiologists who have not only united mathematical ability to other acquirements, but have left the records of their labor in the shape of mathematical formulæ. Some of these formulæ may become useful in studying the cause of organic forms.

For some time the writer has been engaged in collecting materials of the kind described. The preparation of this paper was undertaken from a desire to render useful the labor incurred in the collection; but circumstances have prevented as full a treatment of the subject as we could have wished. Many of the authors cited deserve more attention than we have been able here to give to them, and others have been left unnoticed because our limits did not permit us to speak of them as they deserve. On another occasion we hope to return to our subject under more favorable circumstances.

In submitting this paper to the reader, I have endeavored to make the best selection of matter for general perusal, and to supply some desirable references for the use of students who have paid less attention than myself to the subject.*

In concluding, I desire to express my thanks to several members of the Academy of Natural Sciences who have assisted and encouraged me, and especially to Dr. Jos. Leidy.

A Review of the TERNS of North America.

BY ELLIOTT COUES.

Considerable difference has prevailed among ornithological writers with regard to the relationships of many of the North American *Sterninæ* with the representative species of Europe. Having at command a very extensive series of specimens from both continents, I have instituted a careful comparison of the more or less intimately related species, believing that the results of such an investigation would not prove unacceptable to ornithologists. While this has been the principal aim of the present paper, I have endeavored to present fairly the data tending to determine some other points of synonymy and relationship which even at this late day remain open to discussion; and to give such stages of plumage as are not already too well known to require notice. The paper is not to be considered in any sense as a monograph; I have endeavored to express its character in its title.

I am under particular obligations to Mr. G. N. Lawrence and Mr. D. G. Elliot, for the opportunity of examining several unique and typical specimens, and unusual stages of plumage, of which the museum of the Smithsonian Institution does not contain examples.

Family *LARIDÆ*.

Subfamily STERNINÆ.

Section *STERNEÆ*.

Genus GELOCHELIDON Brehm.

Gelochelidon, Brehm, Vög. Deutsch. 1830. Type *S. anglica*, Mont.
Laropis, Wagler, Isis, 1832, p. 1225. Same type.

CHAR.—Bill shorter than the head, extremely robust, not very acute; its height at base nearly a third of its total length along culmen; prominence at symphysis well marked, but not very acute, situated so far back as to make the gonys equal in length to the rami, reckoning from the termination of the feathers on the side of the mandible. Culmen very convex; gonys straight; commissure gently curved. Wings exceedingly long, and acute; each feather a full inch longer than the next. Tail rather short, contained $2\frac{1}{2}$ times in the wing; in form deeply emarginate, but its lateral feathers without the elongation of *Sterna*. Feet long and stout; tarsus a little shorter than the bill, exceeding the middle toe and claw. Hind toe well developed; inner shorter

* Several authors not mentioned in our former work may here be briefly cited.
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